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Integral invariants in $N = 4$ SYM and the effective action for coincident D-branes

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Abstract: The construction of supersymmetric invariant integrals is discussed in a superspace setting. The formalism is applied to $D = 4$, $N = 4$ SYM and used to construct the F^2, F^4 and $(F^5 + \partial^2 F^4)$ terms in the effective action of coincident D-branes. The results are in agreement with those obtained by other methods. A simple derivation of the abelian $\partial^4 F^4$ invariant is given and generalised to the non-abelian case. We also find some double-trace invariants. The invariants are interpreted in terms of superconformal multiplets: the F^2 and F^4 terms are given by one-half BPS multiplets, the $(F^5 + \partial^2 F^4)$ arises as a full superspace integral of the Konishi multiplet K and the abelian $\partial^4 F^4$ term comes from integrating the fourth power of the field strength superfield. Counterparts of the abelian invariants are exhibited for the $D = 6$, $(2, 0)$ tensor multiplet and the $D = 3$, $N = 8$ scalar multiplet. The method is also applied to $D = 4$, $N = 8$ supergravity. All invariants in the linearised theory (with SU(8) symmetry) which arise from partial superspace integrals are constructed.

Keywords: Superstrings and Heterotic Strings, D-branes, Supersymmetric Effective Theories, Superspaces.

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Contents

1. Introduction

An intriguing problem in the theory of D-branes is the question of what is the effective action for a set of coincident branes. In the abelian case the Born-Infeld approximation, involving no derivatives of the field-strength, is well-defined [1], but it is not clear that there is a meaningful generalisation of this to the non-abelian case. In any case, there are also higher derivative corrections to Born-Infeld for a single brane, so that it is perhaps more sensible to consider the general effective action, including all derivative corrections. As yet there is not a well-established principle for finding this although there have been some suggestions as to what it might be. The non-abelian version of the bosonic Born-Infeld action, constructed using the symmetrised trace, was proposed in [2], but it is known from string theory calculations that this is incomplete [3]. An attempt was made via modified κ-symmetry transformations in [4], but this does not seem to agree with string theory [5]. A constructive principle, based on the theory admitting solutions of a particular type has been put forward in [6]. This correctly reproduces the abelian Born-Infeld action [7] and is so far in agreement with known results from string theory in the non-abelian case [8]. Although it is difficult to compute the full effective action in a closed form, calculations so far have been carried out to order α'^4 [9]. Further results on derivative corrections have been obtained in [10, 11].

The abelian Born-Infeld approximation is also known to be determined completely by supersymmetry. The κ-symmetric extension of the Born-Infeld action for a D-brane was discussed in references [12]–[14], but perhaps the fact that supersymmetry determines the action completely is made clearer in the superembedding formalism [15]. In this approach there is a natural constraint on any superembedding which has a simple geometrical interpretation and which determines the lowest-order non-linear field equations of most single branes in type II string theory and M-theory uniquely [16, 18]; it also explains the structure of κ-symmetry transformations. Starting from the Wess-Zumino term one can construct the action, including the Dirac-Born-Infeld term in the case of D-branes, systematically [17]. The exceptional cases are the branes with low codimension, but here one can argue, for example in codimension zero [19], that the multiplet structure of the theory yields a unique set of constraints which leads to the κ-symmetric Born-Infeld action. However, it is not clear what the non-abelian generalisation of this is, nor is it clear how higher-derivative terms in the abelian effective action arise, although some progress has been made recently [20]. In [21] a study was made of a toy model which was designed to represent a set of coincident space-filling branes in three dimensions. A number of non-abelian extensions of the abelian super Born-Infeld theory were found all of which incorporate Tseytlin's action. However, there seems to be no obvious way of selecting out a unique action. This type of approach was also advocated for D0-branes in [22] and [23]

In the superembedding approach for a single brane the target space supersymmetry is manifest, and one uses the modified field strength $\mathcal{F}$ which satisfies the Bianchi identity $d\mathcal{F} = H$ where H is the pull-back of the NS three-form. This formalism is not related in a simple way to the usual $N = 1$, $D = 10$ superspace version of the deformed super Maxwell theory, but it is possible to derive the $D = 10$, $N = 1$ superspace constraints corresponding to the abelian D9-brane action in a systematic manner. This was done to order α'^4 in [24], but it does not seem to be straightforward to adapt them to the non-abelian case.

Another way of looking at this problem is provided by spinorial cohomology. This is a superspace cohomology related to pure spinors. The relevance of such spinors to supersymmetric field theories in ten and eleven dimensions was pointed out in [25, 26]. Essentially one considers forms with only spinorial indices (which are therefore symmetric multi-spinors) with the gamma-traces removed.[1] One then forms a derivative by acting on such an object with the super-covariant derivative D_α and projecting out the gamma-trace. In this way one arrives at a complex and its associated cohomology [27]. This cohomology is isomorphic to pure spinor cohomology [28]. Deformations of $D = 10$ super Yang-Mills theory are given by elements of the second spinorial cohomology group with physical coefficients. It is easy enough to find the first deformation (corresponding to F^4) [29, 30] but the analysis at higher orders in α' is more difficult, although α'^3 has been studied in the abelian case [31].

[1] By this we mean that, if one contracts on any pair of spinor indices with a single gamma matrix, one gets zero.

In some recent papers higher-order actions for $D = 10$ super Yang-Mills theory have been constructed, to quadratic order in the fermions, using supersymmetry (in components) and the Noether method. The terms that have been built include the F^4 action, the F^5 action, absent in the abelian case, and most recently a higher derivative abelian $\partial^4 F^4$ term [32, 33]. The F^5 term has also been discussed in $D = 4$, $N = 4$ Yang-Mills theory formulated in $N = 1$ superspace [34, 35]. These results are in agreement with those of [6]. In [9] the method of [6] was used to construct the purely bosonic terms at $(\alpha')^4$ in the non-abelian theory; the result obtained there incorporates the above $\partial^4 F^4$ term in the abelian limit.

In this article we shall rederive these terms in a simple manner in four dimensions. Four dimensions is easier to work with than ten because one can make use of harmonic superspace techniques to construct integrals which involve fewer than the maximum number of odd coordinates but which are still manifestly supersymmetric.[2] It turns out that the known terms can all be interpreted in terms of $N = 4$ superconformal multiplets. The usual Yang-Mills action is a component of the supercurrent multiplet, the F^4 term comes from a series C one-half BPS multiplet with dimension 4 and the F^5 term comes from a descendant one-quarter BPS state. It can alternatively be expressed as a full superspace integral of the Konishi multiplet, and so is not truly BPS. These are all of the single-trace terms that can arise as integrals over fewer than sixteen odd coordinates, although there are also double-trace one-half BPS and one-quarter BPS terms. There are no true series B integrands (which would have to be at least triple-trace); the only allowed series B integrand is also a descendant of Konishi and gives the same F^5 result. The abelian $\partial^4 F^4$ invariant is a full superspace integral of an integrand of the form W^4 where W is the $N = 4$ field strength superfield whose leading component transforms under the six-dimensional representation of SU(4). This admits two single-trace non-abelian generalisations as well as two double-trace ones. It also generalises to the $D = 6$, $(2, 0)$ tensor multiplet and the $D = 3$, $N = 8$ scalar multiplet, these being the worldvolume multiplets of the M5 and M2 branes respectively. The sub-superspace integral invariants in these theories, and in $D = 4$, $N = 8$ supergravity (section 5), are examples of BPS contributions reviewed in detail in [36].

The invariants we find are constructed from the on-shell field-strength superfield. This means that in general they are not complete. In appendix B we outline a method for finding these completions as well as the modifications to the supersymmetry transformations.

2. Integral invariants

The simplest way to construct a supersymmetric integral invariant is to integrate a superfield over the whole of superspace. However, it is well-known that one can also obtain invariants by integrating over a smaller number of θs; for example, one can integrate chiral superfields over half of the odd coordinates. A systematic investigation of N-extended $D = 4$ Poincaré supersymmetric invariants and the corresponding measures was given in [37] where the invariants were dubbed superactions. The integrands and measures were

[2] The invariants we find are unique and so must be the $N = 4$, $D = 4$ dimensional reductions of $N = 1$, $D = 10$ invariants.

taken to be Lorentz scalars, although they are in general not scalars under the internal U(N) or SU(N) symmetry group. Such measures take a simpler form in harmonic superspace [38, 39]; indeed one can reformulate superactions as harmonic superspace integral invariants where the integrands are scalars up to internal charges [40].

Harmonic superspaces are designed to facilitate the study of generalised forms of chirality, called Grassmann analyticity, or G-analyticity for short. G-analytic superfields can be thought of as depending on fewer odd coordinates than a complete scalar superfield in ordinary superspace. Harmonic superactions therefore take the form of integrals of a superfield obeying some particular G-analyticity constraints (there are several possibilities for general N) matched with the appropriate measure. It should be noted that, although a generic G-analytic superfield will be a full superfield in its dependence on all of the reduced set of odd coordinates, there are a few examples of such superfields which are ultra-short and which can therefore be integrated over even fewer odd coordinates than one might have expected at first sight. This corresponds to the two types of superaction which were studied in [37] in terms of the constraints that the integrands had to satisfy.

The above considerations suggest a way of constructing all possible invariants in a given theory which involve integrating over fewer than the maximal number of odd coordinates. If one knows the multiplets concerned then one can investigate the short ones, which will usually correspond to G-analytic fields on some harmonic superspace. In the $N = 4$ super Yang-Mills theory, the field strength superfield is a scalar superfield W_{ij}, $i, j = 1, \ldots, 4$, transforming under the six-dimensional representation of SU(4). The simplest multiplets one can construct are gauge-invariant products of Ws projected onto irreducible representations of SU(4). These multiplets are actually superconformal multiplets and have been widely studied in the literature. It is therefore a simple task to list the multiplets and to see which ones can be integrated to give integral invariants.

There are also superconformal multiplets which are not Lorentz scalars. However, these can at best be subject to series A-type shortenings which have the form of spinorial divergences (an example is the $D = 4$, $N = 1$ supercurrent $J_{\alpha\dot{\alpha}}$ which obeys the constraints $D^{\alpha}J_{\alpha\dot{\alpha}} = \bar{D}^{\dot{\alpha}}J_{\alpha\dot{\alpha}} = 0$). However, constrained superfields of this type do not give rise to integral invariants. In addition, there are multiplets which are not primary superconformal fields; such fields have leading components involving the derivatives of the field strength. They can be expressed as linear combinations of primaries and descendants of primary fields, or possibly as products of such fields. In either case it seems highly unlikely that such multiplets could give rise to integral invariants involving sub-superspace integrals.

2.1 Superconformal multiplets

Representations of N-extended superconformal symmetry in $D = 4$ are specified by $N + 3$ quantum numbers $(L, R, J_1, J_2, a_1, \ldots, a_{N-1})$, where L is the dilation weight, R is the R-charge, J_1 J_2 are the two spin quantum numbers, and the a_is are the Dynkin labels of an irreducible internal SU(N) representation [41]. The unitary representations have to satisfy certain unitarity bounds which can be one of three types:

$$\begin{aligned}
\text{Series A}: L &\geq 2+2J_2-R+\frac{2m}{N}\,, & L &\geq 2+2J_1+R+2m_1-\frac{2m}{N} & & \\
\text{Series B}: L &= -R+\frac{2m}{N}\,, & L &\geq 1+m_1+J_1\,, & J_2 &= 0 \\
\text{or}: L &= R+2m_1-\frac{2m}{N}\,, & L &\geq 1+m_1+J_2\,, & J_1 &= 0 \\
\text{Series C}: L &= m_1\,, & R &= \frac{2m}{N}-m_1\,, & J_1 &= J_2 = 0\,.
\end{aligned} \tag{2.1}$$

Here m is the total number of boxes in the Young tableau corresponding to the representation $(a_1,\ldots,a_{N-1})$, and m_1 is the number of boxes in the first row. For $N=4$ SYM the superconformal group is $SU(2,2|4)$; representations of this group have $R=0$. For the rest of this section we shall focus on the $N=4$ case. These representations were discussed in the context of the AdS/CFT conjecture in [42].

Series B and C multiplets are always short and there can also be shortened representations in series A, although these do not correspond to G-analytic superfields. The series C multiplets can be either one-half BPS, which means they depend generically on one-half of the odd coordinates, or one-quarter BPS which depend essentially on three-quarters of the odd coordinates. Short multiplets have been discussed in a harmonic superspace framework in [40] and [43]–[46].

The one-half BPS multiplets divide into two cases: the single-trace operators, also known as CPOs, which correspond to the supergravity Kaluza-Klein states in the AdS/CFT correspondence, and multi-trace products of these. The CPO A_k is defined to be the single-trace product of W^k taken in the representation $[0k0]$ of $SU(4)$. The field strength W ($k=1$) is only a superconformal field in the free theory and $A_2 := T$ is the supercurrent multiplet which contains all of the conserved currents of $N=4$ SYM.[3] In the interacting theory both T and A_3 are extra-short, while any $A_k,\ k\geq 4$ is not subject to any additional shortening.

The one-quarter BPS multiplets can be subdivided into two classes as well: the true BPS operators which are protected and which do not have anomalous dimensions, and those which are descendants of long operators. In the quantum theory the latter develop anomalous dimensions and cease to exist as short operators. The true one-quarter BPS operators are at least double-trace and have $SU(4)$ Dynkin labels $[pqp]$, while the descendants can be either single- or multi-trace. A detailed discussion of these operators and their mixing properties is given in [47, 48].

The series B scalar operators can be thought of as one-eighth BPS multiplets which depend on seven-eighths of the odd coordinates. Again there are true multiplets and descendants. The former are at least triple trace while the latter can be single- or multi-trace. These operators have $SU(4)$ Dynkin labels $[q+2,p,q]$ (or the conjugate). The one-quarter BPS multiplets are not subject to extra shortening, but the one-eighth BPS multiplets which saturate both unitarity bounds are subject to a second-order spinorial derivative as well as a first-order G-analyticity constraint.

[3] In ordinary superspace the supercurrent is $\mathrm{tr}(W_{ij}W_{kl}-\frac{1}{24}\epsilon_{ijkl}\epsilon^{mnpq}W_{mn}W_{pq})$.

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There are also some series A operators which are shortened. An example is the square of the supercurrent in the representation [020]. This is subject to a second-order derivative constraint which survives in the quantum theory. Other series A scalar operators are long operators which are entire scalar superfields on ordinary superspace. The simplest example is the Konishi superfield $K := \mathrm{tr}(W^2)$ in the singlet representation of SU(4). In the free theory this superfield satisfies a second-order constraint [49].

2.2 Harmonic superspace

We denote the spinorial derivatives on Minkowski superspace M_N by $(D_{\alpha i}, \bar{D}^j_{\dot\beta})$, $i,j = 1,\ldots,N$, in two-component spinor notation. The supersymmetry algebra is $[D_{\alpha i}, \bar{D}^j_{\dot\beta}] = i\delta_i{}^j\partial_{\alpha\dot\beta}$. We define G-analyticity of type (p,q) to be a set of p Ds and q $\bar{D}$s which mutually anti-commute. The space of G-analyticities of type (p,q) is the coset space $K_{p,q} = S(\mathrm{U}(p) \times \mathrm{U}(N-(p+q) \times \mathrm{U}(q))\backslash \mathrm{SU}(N)$. This is easy to see: let $u_I{}^i \in \mathrm{SU}(N)$, where $\mathrm{SU}(N)$ acts on i to the right and the isotropy group acts on I to the left, and let $(u^{-1})_i{}^I$ denote its inverse. We split the index $I = (r, R, r')$ where $r = 1,\ldots,p$, $R = p+1,\ldots,N-q$, $r' = N-q+1,\ldots,N$; the isotropy group acts in an obvious manner. Now set $D_{\alpha I} := u_I{}^i D_{\alpha i}$ and $\bar{D}^I_{\dot\alpha} := \bar{D}^i_{\dot\alpha}(u^{-1})_i{}^I$; clearly the derivatives $(D_{\alpha r}, \bar{D}^{r'}_{\dot\alpha})$ mutually anti-commute.

We define (N,p,q) harmonic superspace to be $M_N \times K_{p,q}$. A field on this space is equivalent to a field on $M_N \times \mathrm{SU}(N)$ which is equivariant with respect to the isotropy group, i.e its dependence on the coordinates of the isotropy group is fixed. Such a field can be expanded in harmonics on the coset with coefficients which are conventional superfields whence the nomenclature. A G-analytic superfield on this space is one which is annihilated by $(D_{\alpha r}, \bar{D}^{r'}_{\dot\alpha})$; it will therefore depend on $4N - 2(p+q)$ odd coordinates. Fields on (N,p,q) superspace can also be harmonic analytic which means they are holomorphic with respect to the $\bar\partial$ operator on $K_{p,q}$; such superfields have finite harmonic expansions since the coset space is a compact complex manifold, and all of the supermultiplets which arise in the superconformal context are of this type.

For $N = 4$ SYM we are interested in $(p,q) = (2,2)$ for one-half BPS, $(p,q) = (1,1)$ for one-quarter BPS and $(p,q) = (1,0)$ for one-eighth BPS.

(4, 2, 2). On this superspace we split the index $I = (r, r')$, $r = 1,2; r' = 3,4$; the basic G-analytic superfield is

$$W := \frac{1}{2}\epsilon^{rs} u_r{}^i u_s{}^j W_{ij}\,. \tag{2.2}$$

It is annihilated by $D_{\alpha r}, r = 1,2$ and $\bar{D}^{r'}_{\dot\alpha}, r' = 3,4$; it is also harmonic analytic on the coset $S(UU(2) \times \mathrm{U}(2))\backslash \mathrm{SU}(4)$. The CPOs are given by $A_k = \mathrm{tr}(W^k)$. The other one-half BPS multiplets are given by products of the A_ks. Note that these superfields depend on half of the odd coordinates, that is to say, two θs and two $\bar\theta$s. Now the highest power of odd variables in W is two, so T has up to four powers of θ and A_3 up to six powers. These supermultiplets are thus extra short, whereas all the superfields with $k \geq 4$ depend on all four θs and $\bar\theta$s.

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$(4,1,1)$. On this space we split the index $I = (1,r,4)$ where $r \in \{2,3\}$. The basic superfield is

$$W_{1r} := u_1{}^i u_r{}^j W_{ij}\,. \tag{2.3}$$

This is easily seen to be $(1,1)$ analytic, i.e. $D_{\alpha 1} W_{1r} = \bar{D}^4_{\dot\alpha} W_{1r} = 0$. The CPOs are given by single traces of this superfield with the SU(2) indices symmetrised:

$$A_{r_1\ldots r_k} = \mathrm{tr}(W_{1(r_1} \ldots W_{1r_k)})\,. \tag{2.4}$$

The true one-quarter BPS multiplets are given by products of As with at least one contraction and the remaining SU(2) indices symmetrised, e.g. the operator $A_{u(rs}T_{t)}{}^u$. An example of a descendant is the operator $\mathrm{tr}(Y^2)$ where $Y := \epsilon^{rs} W_{1r} W_{1s}$.

$(4,1,0)$. On this space we put $I = (1,r)$ with $r \in \{2,3,4\}$, and define a field W_{1r} in the same way as in the $(1,1)$ case, although it now only satisfies the constraint $D_{\alpha 1} W_{1r} = 0$. Again the CPOs are constructed from single-trace products of this superfield with all of the SU(3) indices symmetrised. To form a true one-eighth BPS operator one has to multiply at least three different As together, contract one (or possibly more) index from each operator with ϵ^{rst} and symmetrise on the remaining indices. The simplest example has dimension six; it is

$$\mathcal{O} := \epsilon^{rst}\epsilon^{uvw} T_{ru} T_{sv} T_{tw}\,. \tag{2.5}$$

This operator corresponds to the SU(4) Dynkin labels [400]. The simplest descendant operator in this class is $\epsilon^{rst}\mathrm{tr}(W_{1r}W_{1s}W_{1t})$ with Dynkin labels [200].

2.2.1 The maximal coset

Instead of using the coset space $K_{p,q} = S(\mathrm{U}(p) \times \mathrm{U}(N-(p+q) \times \mathrm{U}(q))\backslash \mathrm{SU}(N)$ one can replace the middle factor by any subgroup of $\mathrm{U}(N-(p+q))$. Indeed, in some situations it is useful to consider the maximal coset space which is obtained by taking the isotropy group to be the maximal torus, $(\mathrm{U}(1))^3$ for $N=4$ [45, 50]. The coset $K := (\mathrm{U}(1))^3\backslash \mathrm{SU}(4)$ is the space of full flags in $\mathbb{C}^4$. In this case we can write an element of SU(4) as $u_I{}^i = (u_1{}^i, u_2{}^i, u_3{}^i, u_4{}^i)$ and its inverse by $(u^{-1})_i{}^I$. We have three U(1) charges which we can take to be $+1$ for each of the indices $1,2,3$. The index 4 then has charge -1 for each U(1). Upper indices have the opposite charges to lower indices. We can convert SU(4) indices to $\mathrm{U}(1)^3$ indices by means of u and u^{-1}, and differentiation on the coset is carried out using the right-invariant vector fields $D_I{}^J$. The set $\{D_I{}^J | I < J\}$ corresponds to the components of the $\bar\partial$ operator on K while the set $\{D_I{}^J | I > J\}$ corresponds to the components of the conjugate operator ∂. The diagonal derivatives are associated with the isotropy algebra; we can define them in accordance with the above charge assignments. The three $\mathfrak{u}(1)$ generators are $D^{(r)} := D_r{}^r,\ r = 1,2,3$ where there is no sum, and we have

$$\begin{aligned} D^{(r)} u_s{}^i &= \delta_s{}^r u_s{}^i\,, && \text{no sum on } s \\ D^{(r)} u_4{}^i &= -u_4{}^i,\,, && r = 1,2,3\,. \end{aligned} \tag{2.6}$$

The charges carried by the coset space derivatives are then given by their numerical indices so that, for example, $D_1{}^2$ has charge $+1$ with respect to the first U(1)and charge -1 with respect to the second.

The properties of the Yang-Mills field strength superfield W_{ij} are listed in the appendix. In harmonic superspace we define $W_{IJ} = u_I{}^i u_J{}^j W_{ij}$. The G-analyticity conditions satisfied by W are

$$\begin{aligned} \nabla_{\alpha I} W_{IJ} &= 0\,, \qquad \text{no sum on } I \\ \bar{\nabla}^K_{\dot\alpha} W_{IJ} &= 0\,, \qquad \text{if } K \neq I, J\,, \end{aligned} \tag{2.7}$$

where ∇ denotes the superspace gauge-covariant derivative. In this superspace one can therefore work with numerical indices but still retain SU(4) covariance provided that each charge in an invariant vanishes.

2.3 Invariants

To form an invariant we now have to integrate one of the above supermultiplets over the appropriate measure. There is only one special case, the supercurrent T, which is extra short and one-half BPS. It depends on essentially four odd coordinates which suggests that there should be an invariant of the form

$$I_0 = \int d\mu\, T \sim \int d^4x\, \left(F^2 + \cdots\right), \tag{2.8}$$

where the measure is

$$d\mu := d^4x\, du\, [D_3 D_4]^2 \tag{2.9}$$

with $D^2 := \frac{1}{2} D_\alpha D^\alpha$ for any D, and where du denotes the standard Haar measure on the internal coset $K_{2,2}$. This expression is not manifestly supersymmetric; the proof that it is was given in terms of superactions in [37]. There is no integral invariant one can form using A_3 so all of the rest are standard harmonic superspace integrals.

The one-half BPS multiplets can be integrated with respect to the measure

$$d\mu_{2,2} := d^4x\, du\, [D_3 D_4 \bar{D}^1 \bar{D}^2]^2\,. \tag{2.10}$$

However, there is a U(1) factor in the isotropy group and the measure has charge -4 with respect to this group. The only integrands that are allowed will therefore have U(1) charge $+4$, and there are just two possibilities, A_4 and T^2. The first of these, $\int d\mu_{2,2}\, A_4$ gives the F^4 term in the non-abelian Born-Infeld action while the second gives a $(\mathrm{tr}(F^2))^2$ term.

The one-quarter BPS multiplets can be integrated with the measure

$$d\mu_{1,1} := d^4x\, du\, [D_2 D_3 D_4 \bar{D}^1 \bar{D}^2 \bar{D}^3]^2\,. \tag{2.11}$$

There is only one single-trace possibility which has the right charges; it has SU(4) Dynkin labels [202] and can only be realised by the descendant operator $\mathrm{tr}(Y^2)$ discussed above. In fact this operator can be written as

$$\mathrm{tr}(Y^2) = [D_1 \bar{D}^4]^2 K\,, \tag{2.12}$$

where $K := \mathrm{tr}(W_{ij}\bar{W}^{ij})$ is the Konishi superfield. We therefore have

$$I_3 = \int d\mu_{1,1} \mathrm{tr}(Y^2) = \int d^4x\, d^{16}\theta K \sim \int d^4x \left(F^5 + \partial^2 F^4 + \cdots\right). \tag{2.13}$$

There is also a double-trace one-quarter BPS multiplet in the representation [202]; it is given on $(4,1,1)$ superspace by $T_{rs}T^{rs}$. The integral of this should give rise to double-trace $\partial^2 F^4$ terms.

For the one-eighth BPS supermultiplets the measure is

$$d\mu_{1,0} := d^4x\, du\, [D_2 D_3 D_4 \bar{D}^1 \bar{D}^2 \bar{D}^3 \bar{D}^4]^2 . \tag{2.14}$$

There is only one possible integrand, the single-trace descendant $\epsilon^{rst}\mathrm{tr}(W_{1r}W_{1s}W_{1t})$. In fact, it can be written as $(D_1)^2 K$, so that the invariant is just the F^5 one given above.

The above invariants are the only ones that can be constructed in $N = 4$ SYM as integrals involving fewer than sixteen θs. We note that there is no independent F^6 term which confirms that this term, present in Born-Infeld, is generated by the F^4 term, as shown in [33]. This result has also been established directly in four dimensions in the off-shell $N = 3$ superspace formalism [51].

The simplest integrals that can be constructed using the full superspace measure, apart from the one we have discussed, involve four powers of W. If we switch to $SO(6)$ notation and regard the real superfield $W_A, A = 1, \ldots, 6$, (on Minkowski superspace) as a vector under this group, we can write the possible integrands as $\mathrm{symtr}(W_A W_A W_B W_B)$, $\mathrm{tr}([W_A, W_B][W_A, W_B])$, K^2 and $T_{AB}T_{AB}$, where T_{AB} denotes the supercurrent which is a symmetric traceless tensor in this notation. The first of these is the non-abelian version of the $\partial^4 F^4$ term found in [33], the second is a similar term which vanishes in the abelian case and the last two are double-trace expressions which both reduce to the first integrand in the abelian case.

3. F terms

In this section we evaluate the pure F contributions to these integrals using the formulae given in the appendix. It is convenient to use the maximal coset superspace for this. The α'^2 term is well-known, so we shall not give it again here; it is easy to check that it does indeed have the correct Born-Infeld form. We shall also restrict our attention to single-trace integrands.

$\mathcal{O}(\alpha'^3)$. The invariant can be written, using the maximal coset, as

$$I_3 = \int d^4x\, du\, d^{16}\theta\, \mathrm{tr}(W_{12}W_{34}) . \tag{3.1}$$

This expression can easily be seen to be the same as the integral over ordinary superspace of $\mathrm{tr}(W_A W_A)$. However, it is useful to use harmonic notation even for full superspace integrals as it is easier to evaluate them this way.

As usual, we can carry out the integration over the odd coordinates by applying eight Ds and $\bar{D}$s to the integrand. We can substitute these by gauge-covariant spinorial derivatives, as the integrand is gauge-invariant, and so the task is to evaluate

$$\mathrm{tr}\,[\nabla_1\nabla_2\nabla_3\nabla_4\bar{\nabla}^4\bar{\nabla}^3\bar{\nabla}^2\bar{\nabla}^1]^2(W_{12}W_{34})\,, \tag{3.2}$$

in terms of the component fields. The pure field strength contribution is

$$\begin{aligned} I_3(F) = \mathrm{tr}\Big(& 6M_\alpha{}^\beta M_\beta{}^\gamma M_\gamma{}^\alpha \bar{M}_{\dot\alpha}{}^{\dot\beta}\bar{M}_{\dot\beta}{}^{\dot\alpha} - 2M_\alpha{}^\beta M_\beta{}^\gamma \bar{M}_{\dot\alpha}{}^{\dot\beta} M_\gamma{}^\alpha \bar{M}_{\dot\beta}{}^{\dot\alpha} - \\ & - 6\bar{M}_{\dot\alpha}{}^{\dot\beta}\bar{M}_{\dot\beta}{}^{\dot\gamma}\bar{M}_{\dot\gamma}{}^{\dot\alpha} M_\alpha{}^\beta M_\beta{}^\alpha + 2\bar{M}_{\dot\alpha}{}^{\dot\beta}\bar{M}_{\dot\beta}{}^{\dot\gamma} M_\alpha{}^\beta \bar{M}_{\dot\gamma}{}^{\dot\alpha} M_\beta{}^\alpha + \\ & + \bar{M}_{\dot\alpha}{}^{\dot\beta}\bar{M}_{\dot\beta}{}^{\dot\alpha}\nabla_{\gamma\dot\gamma}M_\alpha{}^\beta\nabla^{\gamma\dot\gamma}M_\beta{}^\alpha + M_\alpha{}^\beta \bar{M}_{\dot\alpha}{}^{\dot\beta}\nabla_{\gamma\dot\gamma}M_\beta{}^\alpha\nabla^{\gamma\dot\gamma}\bar{M}_{\dot\beta}{}^{\dot\alpha} + \\ & + M_\alpha{}^\beta \bar{M}_{\dot\alpha}{}^{\dot\beta}\nabla_{\gamma\dot\gamma}\bar{M}_{\dot\beta}{}^{\dot\alpha}\nabla^{\gamma\dot\gamma}M_\beta{}^\alpha + \bar{M}_{\dot\alpha}{}^{\dot\beta} M_\alpha{}^\beta\nabla_{\gamma\dot\gamma}M_\beta{}^\alpha\nabla^{\gamma\dot\gamma}\bar{M}_{\dot\beta}{}^{\dot\alpha} + \\ & + \bar{M}_{\dot\alpha}{}^{\dot\beta} M_\alpha{}^\beta\nabla_{\gamma\dot\gamma}\bar{M}_{\dot\beta}{}^{\dot\alpha}\nabla^{\gamma\dot\gamma}M_\beta{}^\alpha + M_\alpha{}^\beta M_\beta{}^\alpha\nabla_{\gamma\dot\gamma}\bar{M}_{\dot\alpha}{}^{\dot\beta}\nabla^{\gamma\dot\gamma}M_{\dot\beta}{}^{\dot\alpha}\Big) \end{aligned} \tag{3.3}$$

where $M_{\alpha\beta}$ is the field strength tensor in spinor notation, $F_{\alpha\dot\alpha,\beta\dot\beta} = \epsilon_{\dot\alpha\dot\beta}M_{\alpha\beta} - \epsilon_{\alpha\beta}\bar{M}_{\dot\alpha\dot\beta}$. This result agrees with other calculations reported in the literature.

It is easy to see that this vanishes in the abelian limit. In this case, the Konishi multiplet obeys the constraint $D_{ij}K = 0$, $D_{ij} := \frac{1}{2}\epsilon^{\alpha\beta}D_{\alpha i}D_{\beta j}$, and so we have a full superspace integral of a constrained superfield which is trivially zero.

$\mathcal{O}(\alpha'^4)$. At α'^4, in the abelian case, there is an invariant of the form

$$I_4 = \int d^4x\, d^{16}\theta\,(W_A W_A)^2 = \int d^4x\, du\, d^{16}\theta\, W_{12}^2 W_{34}^2\,. \tag{3.4}$$

It is simple to evaluate the pure field strength contribution to this invariant. We find

$$\begin{aligned} I_4(F) = {} & M_{\alpha\beta}M^{\alpha\beta}\partial_{\epsilon\dot\epsilon}\partial_{\eta\dot\eta}\bar{M}_{\dot\alpha\dot\beta}\partial^{\epsilon\dot\epsilon}\partial^{\eta\dot\eta}M^{\dot\alpha\dot\beta} + 4M_{\alpha\beta}\partial_{\epsilon\dot\epsilon}\partial_{\eta\dot\eta}M^{\alpha\beta}\bar{M}_{\dot\alpha\dot\beta}\partial^{\epsilon\dot\epsilon}\partial^{\eta\dot\eta}M^{\dot\alpha\dot\beta} + \\ & + \partial_{\epsilon\dot\epsilon}\partial_{\eta\dot\eta}M_{\alpha\beta}\partial^{\epsilon\dot\epsilon}\partial^{\eta\dot\eta}M^{\alpha\beta}\bar{M}_{\dot\alpha\dot\beta}\bar{M}^{\dot\alpha\dot\beta}\,. \end{aligned} \tag{3.5}$$

This result should be compared with that of [33], where higher order supersymmetric actions for the $N = 1$, $D = 10$ Maxwell multiplet were computed using the Noether procedure. The α'^4 terms computed there fall into two groups. The first group consists of the terms that are required to continue the Born-Infeld invariant up to α'^4 and these are induced by the corrections at α'^2. The second group contains terms of the form ∂^4F^4 and represent the start of a new, independent, invariant. The result (3.5), which is necessarily the start of a new invariant, agrees with the dimensional reduction of the second group of α'^4 terms found in [33] to four dimensions.

This invariant can be generalised immediately to the non-abelian case. The simplest single-trace invariant is

$$I_4 = \int d^4x\, d^{16}\theta\,\mathrm{symtr}(W_AW_AW_BW_B) = \int d^4x\, du\, d^{16}\theta\,\mathrm{symtr}(W_{12}W_{12}W_{34}W_{34})\,. \tag{3.6}$$

To calculate the component description one then has to perform the differentiation. The task is to evaluate

$$\mathrm{symtr}[\nabla_1\nabla_2\nabla_3\nabla_4\bar\nabla^4\bar\nabla^3\bar\nabla^2\bar\nabla^1]^2(W_{12}W_{12}W_{34}W_{34})\,. \tag{3.7}$$

This is straightforward, if a little tedious. The pure field strength contribution is

$$\begin{aligned} I_4(F) = \mathrm{symtr}\Bigg(&-2\bar M_{\dot\gamma\dot\delta}\bar M^{\dot\gamma\dot\delta}\Bigg(2[[M_{\gamma\delta},M_\beta{}^\gamma],M^\delta{}_\alpha]M^{\alpha\beta}-[\nabla_{\beta\dot\beta}M_{\gamma\delta},\nabla_\alpha{}^{\dot\beta}M^{\gamma\delta}]M^{\alpha\beta}-\\ &\qquad -5[\nabla_{\beta\dot\beta}M_{\gamma\delta},M^\delta{}_\alpha]\nabla^{\alpha\dot\beta}M^{\beta\gamma}+[M_{\beta\delta},M_\alpha{}^\delta][M_\gamma{}^\beta,M^{\gamma\alpha}]-\\ &\qquad -2[M_{\beta\delta},M_{\alpha\gamma}][M^{\gamma\delta},M^{\alpha\beta}]+\frac12\nabla_{\alpha\dot\alpha}\nabla_{\beta\dot\beta}M_{\gamma\delta}\nabla^{\alpha\dot\alpha}\nabla^{\beta\dot\beta}M^{\gamma\delta}\Bigg)-\\ &-4\bar M_{\dot\delta}{}^{\dot\gamma}M_{\gamma\delta}\Bigg(-2[[\bar M^{\dot\delta}{}_{\dot\gamma},M_\beta{}^\gamma],M_\alpha{}^\delta]M^{\alpha\beta}+2[\nabla_{\beta\dot\beta}\bar M^{\dot\delta}{}_{\dot\gamma},\nabla_\alpha{}^{\dot\beta}M^{\gamma\delta}]M^{\alpha\beta}+\\ &\qquad +4[\nabla_{\beta\dot\beta}\bar M^{\dot\delta}{}_{\dot\gamma},M_\alpha{}^\delta]\nabla^{\alpha\dot\beta}M^{\beta\gamma}+[M_\beta{}^\delta,M_\alpha{}^\gamma][\bar M^{\dot\delta}{}_{\dot\gamma},M^{\alpha\beta}]+\\ &\qquad +[\bar M^{\dot\delta}{}_{\dot\gamma},\nabla_{\alpha\dot\alpha}M_\beta{}^\gamma]\nabla^{\alpha\dot\alpha}M^{\beta\delta}-2[\bar M^{\dot\delta}{}_{\dot\beta},M_\beta{}^\delta][M^{\beta\gamma},\bar M_{\dot\gamma}{}^{\dot\beta}]-\\ &\qquad -\frac12\nabla_{\alpha\dot\alpha}\nabla_{\beta\dot\beta}\bar M^{\dot\delta}{}_{\dot\gamma}\nabla^{\alpha\dot\alpha}\nabla^{\beta\dot\beta}M^{\gamma\delta}\Bigg)+\\ &+2M_{\gamma\delta}M^{\gamma\delta}[\nabla_{\beta\dot\beta}\bar M_{\dot\gamma\dot\delta},\nabla_\alpha{}^{\dot\beta}\bar M^{\dot\gamma\dot\delta}]M^{\alpha\beta}\Bigg)+\text{ conjugate}\,. \end{aligned} \tag{3.8}$$

There is a second invariant which vanishes in the abelian limit; it is

$$I_4'=\int d^4x\,d^{16}\theta\,\mathrm{tr}([W_A,W_B][W_A,W_B])=\int d^4x\,du\,d^{16}\theta\,\mathrm{tr}([W_{12},W_{34}][W_{12},W_{34}])\,. \tag{3.9}$$

We believe that I_4 and I_4' should correspond to the terms $L_{4,2}$ and $L_{4,4}$ given in [9], although we have not checked this in detail. However, $N=4$ supersymmetry by itself does not fix the relative coefficient between the two terms, in contrast to the approach of [9].

At higher powers in α' there will be more and more terms that can be written down. We shall not attempt to classify these. There are, however, only a few which arise from powers of W which give rise to pure F component lagrangians. We briefly discuss two of these.

$\mathcal{O}(\alpha'^5)$. The simplest invariant at this order is

$$I_5=\int d^4x\,du\,d^{16}\theta\,\mathrm{symtr}(W_{12}^3W_{34}^3)\,. \tag{3.10}$$

After a little work the pure field strength contribution can be extracted from this. It is

$$\begin{aligned} I_5(F) = \mathrm{symtr}(&\bar M_{\dot\gamma\dot\delta}\bar M^{\dot\gamma\dot\delta}M_{\gamma\delta}\bar M_{\dot\alpha\dot\beta}\bar M^{\dot\alpha\dot\beta}[M^\delta{}_\beta,M^{\beta\gamma}]-M_{\gamma\delta}M^{\gamma\delta}\bar M_{\dot\gamma\dot\delta}M_{\alpha\beta}M^{\alpha\beta}[\bar M^{\dot\delta}{}_{\dot\beta},\bar M^{\dot\beta\dot\gamma}]+\\ &+8M^{\alpha\beta}\bar M^{\dot\alpha\dot\beta}\nabla_{\alpha\dot\alpha}M_{\gamma\delta}M^{\gamma\delta}\nabla_{\beta\dot\beta}\bar M_{\dot\gamma\dot\delta}\bar M^{\dot\gamma\dot\delta})\,. \end{aligned} \tag{3.11}$$

We observe that this does not vanish in the abelian limit and so provides an example of an invariant of the form ∂^2F^6. Such terms are claimed to absent in the abelian effective action [33], but this one seems to be allowed by four-dimensional $N=4$ supersymmetry.

$\mathcal{O}(\alpha'^6)$. At order α'^6 there is an invariant of the form

$$I_6 = \int d^4x\, du\, d^{16}\theta\, \mathrm{symtr}(W_{12}^4 W_{34}^4) \tag{3.12}$$

which has as its pure field strength part

$$I_6(F) = \mathrm{symtr}(M_{\alpha\beta}M^{\alpha\beta}\bar{M}_{\dot{\alpha}\dot{\beta}}\bar{M}^{\dot{\alpha}\dot{\beta}})^2\,. \tag{3.13}$$

Note that this F^8 term is not the same as the Born-Infeld F^8 contribution. The latter is

$$\frac{1}{2}\left(M^2\right)^3\bar{M}^2 + \frac{15}{8}M^2M^2\bar{M}^2\bar{M}^2 - \frac{1}{2}M^2\left(\bar{M}^2\right)^3\,. \tag{3.14}$$

At orders beyond α'^6 there are no invariants constructed just from integrals of products of Ws which contribute to the pure field strength part of the action.

4. Other models with sixteen supersymmetries

The formalism described above can easily be applied to other models with sixteen supersymmetries such the $D = 3$, $N = 8$ scalar multiplet, the worldvolume multiplet of the M2-brane, and the $D = 6$, $(2,0)$ tensor multiplet, the worldvolume multiplet of the M5-brane. In these cases we do not know what the non-abelian theories are but we can write down some abelian invariants. Both multiplets can be described by Lorentz scalar analytic superfields W on appropriate harmonic superspaces [52]. These superspaces are a little more complicated to describe than the four-dimensional ones because the internal symmetry groups are orthogonal rather than unitary; details of these superpaces and superconformal fields on them can be found in the literature [52]–[55].

As in the $N = 4$ Maxwell case there are only two invariants which can be constructed by integrating over fewer than sixteen odd coordinates. These are, schematically, $d^4\theta\, W^2$ and $d^8\theta\, W^4$. The first of these corresponds to the linearised on-shell action, while the second is the four-field contribution to the brane action which is of generalised Born-Infeld type for the five-brane. The quadratic terms vanish on-shell at the linearised level. These BPS type terms, together with higher-order terms which are required by consistency with supersymmetry, will give rise to the known dynamics of these branes. In fact, for the 5-brane, there is no Lorentz covariant action involving just the fields of the multiplet due to the self-duality of the three-form field strength tensor.[4]. This holds for the non-linear Born-Infeld type theory, but it seems that it is only the F^2 term that has this problem if we try to expand the action in powers of F.

For both multiplets the first non-trivial corrections to the known brane dynamics are therefore given, at the linearised level, by integrals of the form $d^{16}\theta W^4$. For the five-brane this again gives a term of the form $\partial^4F^4 + \cdots$, where F is the three-form field strength of the tensor multiplet, while in the membrane case the bosonic part of the invariant is a quartic expression in the extrinsic curvature. A complete analysis of the latter in the non-linear theory is in preparation [57].

[4]Such an action can be written with the aid of an additional scalar field and gauge invariance [56].

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5. $N = 8$ invariants

In this section we briefly discuss the integral invariants in $N = 8$ supergravity that can be constructed form the linearised field strength W_{ijkl}, $i = 1, \ldots, 8$, which are SU(8) invariant and which are integrals over fewer than thirty-two odd coordinates. The superfield W_{ijkl} is totally antisymmetric and transforms under the seventy-dimensional real representation of SU(8). It obeys the constraints

$$\begin{aligned} \bar{W}^{ijkl} &= \frac{1}{4!}\epsilon^{ijklmnpq}W_{mnpq} \\ D_{\alpha i}W_{jklm} &= D_{\alpha[i}W_{jklm]} \\ \bar{D}^i_{\dot{\alpha}}W_{jklm} &= -\frac{4}{5}\delta^i_{[j}\bar{D}^n_{\dot{\alpha}}W_{klm]n} \end{aligned} \quad (5.1)$$

the third of which follows from the other two. This superfield defines an ultra-short superconformal multiplet. It can be represented on various harmonic superspaces in a similar manner to the $N = 4$ SYM field strength superfield.

We can again construct superconformal multiplets by taking products of W projected into irreducible representations of SU(8). Although the $N = 8$ superconformal group has an R generator, these representations have $R = 0$ because W itself carries no R charge. It turns out that the only series C BPS multiplets that give rise to invariant integrals can be written on $(8, p, p)$ superspace with $p \leq 4$. The only non-zero quantum numbers are the SU(8) Dynkin labels $a_p = a_{8-p}$, a_4 and the dilation weight $L = 2a_p$. Moreover, because of the structure of the measure, all of the integrands have $L = 4$; only for $p = 4$ is $a_4 \neq 0$.

$(8, 4, 4)$. The index I is split into two blocks of four; the superfield can be written

$$W_{1234} := u_1{}^i u_2{}^j u_3{}^k u_4{}^l W_{ijkl} \,. \quad (5.2)$$

It is a singlet under both SU(4)s. The invariant is

$$I_3 = \int d\mu_{4,4}(W_{1234})^4 \,, \quad (5.3)$$

where

$$d\mu_{4,4} := d^4x\, du\, [D_5D_6D_7D_8\bar{D}^1\bar{D}^2\bar{D}^3\bar{D}^4]^2 \,. \quad (5.4)$$

This is the well-known three-loop counterterm [58, 37, 40]. If we carry out the odd integrations we find the supersymmetric completion of the square of the Bel-Robinson tensor.

$(8, 3, 3)$. In this superspace we split $I = (r, R, r')$ where $r \in \{1, 2, 3\}$; $r' \in \{6, 7, 8\}$; $R \in \{4, 5\}$. The superfield is

$$W_{123R} := u_1{}^i u_2{}^j u_3{}^k u_R{}^l W_{ijkl} \,. \quad (5.5)$$

The SU(8) representation is [0020200] and the integrand should be of the form W^4. However, it is easy to see that no such term can be constructed which is invariant under the central SU(2).

$(8,2,2)$. In this case the central index $R \in \{3,4,5,6\}$ and the superfield is

$$W_{12RS} := u_1{}^i u_2{}^j u_R{}^k u_S{}^l W_{ijkl}\,. \tag{5.6}$$

In this case we can form the invariant

$$I_5 = \int d\mu_{2,2} \left(\epsilon^{RSTU} W_{12RS} W_{12TU}\right)^2 \sim \int d^4x\, \partial^4 R^4\,, \tag{5.7}$$

where

$$d\mu_{2,2} := d^4x\, du\, [D_3 D_4 D_5 D_6 D_7 D_8 \bar{D}^1 \bar{D}^2 \bar{D}^3 \bar{D}^4 \bar{D}^5 \bar{D}^6]^2\,. \tag{5.8}$$

$(8,1,1)$. In this space the central index R runs from 2 to 6 and the superfield is

$$W_{1RST} := u_1{}^i u_R{}^j u_S{}^k u_T{}^l W_{ijkl}\,. \tag{5.9}$$

The invariant is

$$I_6 = \int d\mu_{1,1} X_R{}^S X_S{}^R \sim \int d^4x\, \partial^6 R^4\,, \tag{5.10}$$

where

$$X_R{}^S := \epsilon^{ST_1\ldots T_5} W_{RT_1T_2} W_{T_3T_4T_5} \tag{5.11}$$

and where the measure is defined in the obvious way.

These are all the invariants that can be constructed from series C BPS multiplets, but there is also a series B candidate. These multiplets can be realised on $(8,p,0)$ superspace, where $p \leq 4$. In order to be scalars under the internal group SU(p) they can only have $a_p \neq 0$. The only possibility has $p = 2$ and $L = 3$. The superfield is W_{12rs} where $r,s \in \{3,\ldots,8\}$ and the putative invariant is

$$I = \int d\mu_{2,0}\, \epsilon^{r_1\ldots r_6} W_{12r_1r_2} W_{12r_3r_4} W_{12r_5r_6}\,. \tag{5.12}$$

However, the integral turns out to vanish. This is easy to see from representation theory. The multiplet given by the integrand satisfies both of the series B unitarity bounds. This means that it satisfies a first-order G-analyticity D-constraint and a second-order $\bar{D}$ constraint. Since we are integrating over all of the $\bar{\theta}$s it therefore follows that we must get zero.

There are therefore just three $N = 8$ superinvariants which involve sub-superspace integrals. In the context of quantum supergravity they can be interpreted as possible four-point counterterms at three, five and six loop order respectively. It has been argued that the coefficient of the three-loop counterterm vanishes, and there is also a question mark concerning the coefficient of the five-loop counterterm [59]. In [60] it was pointed out that the vanishing of these coefficients can probably be explained be the existence of an off-shell version of $N = 8$ supergravity with more than half of the supersymmetries made manifest. If one can quantise with manifest $N = 6$ supersymmetry, then the onset of divergences would be expected to arise at five loops, while if one can maintain $N = 7$ supersymmetry it should occur at six loops.

These invariants can also be interpreted as dimensional reductions of higher-order terms in the effective actions of type II string theories or M-theory.

6. Conclusions

In this paper we have used superspace methods to construct integral invariants in super Yang-Mills theory and linearised supergravity. These are manifestly supersymmetric in terms of the original on-shell supersymmetry, but the non-linearities which arise as a consequence of including higher-order terms in the action can be computed systematically, at least in principle.

In the Yang-Mills case we have reproduced the string tree-level terms up to order α'^4 which have been found by other means. The terms up to order α'^3 and the abelian α'^4 term are in agreement with other calculations. The non-abelian α'^4 terms seem to be in agreement with those of [9] although we have not checked this in full detail. In addition we have two independent terms whereas the authors of this paper find definite relative coefficients. We have not presented any fermionic terms in this paper, but it should be straightforward, if tedious, to construct the complete component actions from our results.

In the abelian theory it seems possible that the Born-Infeld terms could be generated by the original action and the first F^4 deformation by the linear supersymmetry. For example, as we have seen, there are no independent invariants corresponding to the Born-Infeld F^6 and F^8 terms, although there is a different F^8 invariant. In addition, the relative coefficient between the F^2 term and the rest can be adjusted by rescaling F. If this were to be true we could regard the supersymmetric BI action as the BPS part of the full effective action for a single brane. It would be tempting to try to extend this definition to the non-abelian case: the non-abelian Born-Infeld action should then be the action generated by the two single-trace BPS contributions, F^2 and F^4. However, it would be more difficult to define the higher-order terms unambiguously in this case because the rejection of terms other than slowly-varying ones is no longer valid. It is possible that a criterion for accepting higher-order terms could be devised using group-theoretical structures, which could lead, for example, to the identification of the non-abelian BI sector of the theory with the supersymmetric extension of the Tseytlin symmetrised trace action. Another possibility is that the second ($D = 10$) supersymmetry could play a more significant rôle. Apart from the BPS invariants the other terms involving higher derivatives or commutators are, as we have seen, associated with long multiplets. Without further input the method we have used here would seem to allow a proliferation of such terms at higher order in α'. It is possible that some simplifications might be obtained if we employed the second supersymmetry, although this is something we have not attempted to do here.

We remark that our interpretation of our results is that they should be thought of as arising by dimensional reduction from the effective action of D9-branes or directly from the effective action of D3-branes, in both cases in flat backgrounds. The fact that our results come out rather naturally in terms of superconformal multiplets suggests that it might be useful to consider D3-branes in an $AdS_5 \times S^5$ background, in which case one would expect to have $N = 4$ superconformal symmetry of the action itself.

Although we have not taken all possible symmetries into account it is still possible for us to make a few remarks about some of the conjectures and observations made in [33]. For example, the fact that the three-point function of the open string vanishes can easily be understood in superspace. There are no sub-integrals of this form, and one cannot form a Lorentz and SU(4) invariant from any combination of three fields in the Maxwell multiplet. On the other hand, it is possible to construct odd-point invariants and invariants which do not have any purely bosonic contributions. This can be done directly in $D = 10$ superspace where the superfield Λ^α obeys $D\Lambda \sim F$. An example of both of these types of behaviour is given by $\int d^{10}x\, d^{16}\theta\, \Lambda^{16}(\partial_a\Lambda\gamma_c\partial_b\Lambda)\partial^c F^{ab}$. This has nineteen fields and gives rise to a spacetime invariant in which each term has at least two fermions. However, this does not mean that such terms should arise in practice. For example, it may well be the case that all the integrands are given by superconformal multiplets (in four dimensions), and that, in the abelian theory at least, they could be invariant under an additional "bonus" $\mathrm{U}(1)_Y$ symmetry inherited from IIB supergravity [61]. If true, this might rule out the type of terms we have just mentioned.

We have also seen in the non-abelian case that there are sub-superspace invariants involving double traces which should correspond to string contributions starting at one loop. There are just two such BPS terms, one at α'^2 and one at α'^3, and there are no other BPS integral invariants at higher loops.

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A. $N = 4$ SYM in superspace

We summarise here our conventions for $N = 4$ SYM in superspace [62]. We use a time-favoured metric and convert from vector indices to two-component spinor indices by means of the sigma matrices, e.g. $v_{\alpha\dot\alpha} = (\sigma^a)_{\alpha\dot\alpha}v_a$ where $(\sigma^a)_{\alpha\dot\alpha} = (\sqrt{2})^{-1}(1, i\tau^i)$ where τ^i denotes the usual Pauli matrices. The square root factor means that $u^a v_a = u^{\alpha\dot\alpha}v_{\alpha\dot\alpha}$, or, equivalently, $\eta_{\alpha\dot\alpha,\beta\dot\beta} = \epsilon_{\alpha\beta}\epsilon_{\dot\alpha\dot\beta}$. Our convention for the epsilon tensors is that they are all the same numerically, e.g. $\epsilon_{12} = 1$, so that $\epsilon_{\alpha\beta}\epsilon^{\alpha\gamma} = \delta_\beta{}^\gamma$.

We suppose the gauge group is SU(n). A p-form ϕ in the adjoint representation transforms according to the rule

$$\phi \mapsto g\phi g^{-1}, \qquad \text{where } g \in G \tag{A.1}$$

while the connection A transforms as

$$A \mapsto gAg^{-1} + dgg^{-1}\,. \tag{A.2}$$

The covariant exterior derivative D acts on ϕ by

$$D\phi = d\phi - (-1)^p A\phi + \phi A \tag{A.3}$$

from which

$$D^2\phi = [\phi, F]\,, \qquad \text{where } F = dA + A^2 \tag{A.4}$$

or, in indices, for a scalar ϕ

$$[\nabla_A, \nabla_B]\phi = -t_{AB}{}^C \nabla_C \phi + [\phi, F_{AB}]\,, \tag{A.5}$$

where $t_{AB}{}^C$ is the flat torsion which is zero except for $t_{\alpha i \dot\beta}{}^{j\,c} = -i\delta_i{}^j (\sigma^c)_{\alpha\dot\beta}$. F is Lie algebra valued, and so is skew-hermitean $F = -F^*$.

The constraints on F_{AB} are

$$F_{\alpha i \beta j} = \epsilon_{\alpha\beta} W_{ij}; \qquad F_{\alpha i \dot\beta}{}^{j} = 0 \tag{A.6}$$

which implies that $F^{ij}_{\dot\alpha\dot\beta} = \epsilon_{\dot\alpha\dot\beta}\bar W^{ij}$, where the bar denotes hermitean conjugation. We also impose the self-duality constraint

$$\bar W^{ij} = \frac{1}{2}\epsilon^{ijkl} W_{kl}\,. \tag{A.7}$$

One now employs the Bianchi Identities to find the other components of F_{AB} and the contents of the superfield W_{ij}. We find

$$\begin{aligned} F_{\alpha i, \beta\dot\beta} &= -i\epsilon_{\alpha\beta}\bar\Lambda_{\dot\beta i} \qquad F^i_{\dot\alpha, \beta\dot\beta} = -i\epsilon_{\dot\alpha\dot\beta}\Lambda^i_\beta \\ F_{\alpha\dot\alpha, \beta\dot\beta} &= \epsilon_{\dot\alpha\dot\beta} M_{\alpha\beta} - \epsilon_{\alpha\beta}\bar M_{\dot\alpha\dot\beta}\,, \end{aligned} \tag{A.8}$$

where the second equation expresses the spacetime field strength in terms of a symmetric bi-spinor. We then have the following relations for the derivatives of the superfield W_{ij},

$$\begin{aligned} \nabla_{\alpha i} W_{jk} &= \epsilon_{ijkl}\Lambda^l_\alpha & \bar\nabla^i_{\dot\alpha}\bar W^{jk} &= \epsilon^{ijkl}\bar\Lambda_{\dot\alpha l} \\ \bar\nabla^i_{\dot\alpha} W_{jk} &= 2\delta^i_{[j}\bar\Lambda_{\dot\alpha k]} & \nabla_{\alpha i}\bar W^{jk} &= 2\delta_i^{[j}\Lambda^{k]}_\alpha \\ \nabla_{\alpha i}\bar\Lambda_{\dot\beta j} &= i\nabla_{\alpha\dot\beta} W_{ij} & \bar\nabla^i_{\dot\alpha}\Lambda^j_\beta &= i\nabla_{\beta\dot\alpha}\bar W^{ij} \\ \nabla_{(\alpha i}\Lambda^j_{\beta)} &= -\delta^j_i M_{\alpha\beta} & \bar\nabla^i_{(\dot\alpha}\bar\Lambda_{\dot\beta)j} &= \delta^i_j \bar M_{\dot\alpha\dot\beta} \\ \nabla_{\alpha i}\Lambda^{\alpha j} &= [W_{ik}, \bar W^{jk}] & \nabla_{\alpha i}\nabla^\alpha_j \bar W^{kl} &= 2\delta^{[k}_j [W_{im}, \bar W^{l]m}] \\ \bar\nabla^i_{\dot\alpha}\bar\Lambda^{\dot\alpha}_j &= -[W_{jk}, \bar W^{ik}] & \bar\nabla^i_{\dot\alpha}\bar\nabla^{\dot\alpha j} W_{kl} &= -2\delta^j_{[k}[W_{l]m}, \bar W^{im}] \\ \nabla_{\alpha i} M_{\beta\gamma} &= -i\epsilon_{\alpha(\beta}\nabla_{\gamma)\dot\gamma}\bar\Lambda^{\dot\gamma}_i & \bar\nabla^i_{\dot\alpha}\bar M_{\dot\beta\dot\gamma} &= i\epsilon_{\dot\alpha(\dot\beta}\nabla_{\dot\gamma)\gamma}\Lambda^{\gamma i} \\ \bar\nabla^i_{\dot\alpha} M_{\beta\gamma} &= -i\nabla_{\dot\alpha(\beta}\Lambda^i_{\gamma)} & \nabla_{\alpha i}\bar M_{\dot\beta\dot\gamma} &= i\nabla_{\alpha(\dot\beta}\bar\Lambda_{\dot\gamma)i} \end{aligned} \tag{A.9}$$

From these relations it is clear that the only independent component fields in W_{ij} are the physical fields of the $N = 4$ SYM multiplet, and so they must obey their equations of motion. For example, the spinor equation of motion is

$$\nabla_{\alpha\dot\beta}\bar\Lambda^{\dot\beta}_i = i[W_{ij}, \Lambda^j_\alpha] \qquad \nabla_{\dot\alpha\beta}\Lambda^{\beta i} = i[\bar W^{ij}, \bar\Lambda_{\dot\alpha j}]\,. \tag{A.10}$$

By differentiating this we find the scalar equation of motion

$$\nabla_a\nabla^a W_{ij} = \epsilon_{ijkl}[\Lambda_\alpha{}^k, \Lambda^{\alpha l}] + 2[\bar\Lambda_{\dot\alpha i}, \bar\Lambda^{\dot\alpha}_j] + [W_{ik}, [W_{jl}, \bar W^{kl}]] \tag{A.11}$$

and the vector equation of motion

$$\nabla_{\dot\alpha}{}^\beta M_{\alpha\beta} = i[\Lambda^i_\alpha, \bar\Lambda_{\dot\alpha i}] + \frac{1}{8}([\nabla_{\dot\alpha\alpha} W_{ij}, \bar W^{ij}] - [W_{ij}, \nabla_{\dot\alpha\alpha}\bar W^{ij}])\,. \tag{A.12}$$

B. Supersymmetry transformations

The integral invariants we have described in the text are constructed in terms of the on-shell field strength superfield. However, we know that if we add in higher-order corrections to the standard SYM action the supersymmetry transformations will be modified. As we do not know an off-shell version of this theory, at least with four off-shell supersymmetries, we could in principle take this into account by modifying the constraints on the superspace field strength F_{AB}. As we mentioned in the main text such an approach has been widely discussed in ten dimensions. In this section we sketch an alternative approach to the problem which in principle allows one to construct the full action (up to a given order in α') starting from on-shell invariants of the type we have found. The method, a straightforward generalisation of the BV formalism, allows one both to complete the action, i.e. to find higher-order terms induced from lower-order corrections, to find the amended supersymmetry transformations and to verify that one still has a closed algebra. Alternatively, one could use the Noether procedure [33].

The idea is the following. Starting from an integral invariant one can work out the corresponding expression as a spacetime integral of component fields. One then examines the supersymmetry transformations of the component action constructed by summing the invariants. Since we have an on-shell theory the supersymmetry transformations of the zeroth-order theory only close modulo the field equations and gauge transformations. Moreover, these transformations are non-linear. To deal with this systematically one can use the BRST/BV formalism. This requires the introduction of ghosts for the gauge symmetry and supersymmetry (the latter are constant because the supersymmetry is rigid). In addition, anti-fields for both the physical fields and the ghosts must be introduced. The anti-fields for the fields have ghost number -1 while the anti-fields for the ghosts have ghost number -2. The problem is now to construct an extended (spacetime) action S which satisfies the master equation. If we denote all the fields and the ghosts together by ϕ^i and the corresponding anti-fields by ϕ^*_i we can define an anti-bracket (A, B) of two functionals by[5]

$$(A, B) = \frac{\delta A}{\delta \phi^i}\frac{\delta B}{\delta \phi^*_i} + \frac{\delta B}{\delta \phi^i}\frac{\delta A}{\delta \phi^*_i}\,. \tag{B.1}$$

This can be viewed as an anti-Poisson bracket related to the anti-symplectic two-form $\delta\phi^i \wedge \delta\phi^*_i$. (It is called anti-symplectic because it is a Grassmann odd two-form.) The master equation is

$$(S, S) = 0\,. \tag{B.2}$$

We expand S in powers of α',

$$S = S_0 + S_2 + S_3 + \cdots\,, \tag{B.3}$$

where S_0 is the (extended) classical action, S_2 is the extended action corresponding to the F^4 term and so on. The extended classical action has the form

$$S_0 = S_0^0 + S_0^1 + \mathcal{O}((\phi^*)^2) = S_{cl} + \phi^*_i s\phi^i + \mathcal{O}((\phi^*)^2)\,, \tag{B.4}$$

[5]In this appendix the summation convention is understood to include a spacetime integral.

where s denotes the BRST variation of a field or ghost, and where the superscript counts the number of anti-fields; S_{cl} is the classical action. This implies that the linear terms in ϕ^* at each order encode the modified supersymmetry transformations.

The procedure is now to examine the master equation order by order in α'. At zeroth order we have $(S_0, S_0) = 0$. This equation has been solved, so we can move on to second order where we find $(S_0, S_2) = 0$. The leading term of this equation is

$$\frac{\delta S_{cl}}{\delta \phi^i} \frac{\delta S_2^1}{\delta \phi_i^*} + \frac{\delta S_2^0}{\delta \phi^i} \frac{\delta S_0^1}{\delta \phi_i^*} = 0 \,. \tag{B.5}$$

The second term here is the zeroth order variation of the α'^2 action which we know is an on-shell invariant. So this term has the form $X^i \frac{\delta S_{cl}}{\delta \phi^i}$. Hence we can solve this equation by taking $\frac{\delta S_2^1}{\delta \phi_i^*} = -X^i$. This determines the modified supersymmetry transformations to second order. Assuming we can complete the solution of the master equation at this order we can move on to third order where the situation is very similar. However, at fourth order we find

$$2(S_0, S_4) + (S_2, S_2) = 0 \,. \tag{B.6}$$

Applying $(S_0, .)$ to this equation and using the Jacobi identity we find $(S_0, (S_2, S_2)) = 0$. Assuming that this has a solution of the form $(S_2, S_2) = 2(S_0, Y_4)$, i.e. (S_2, S_2) is cohomologically trivial, we shall then again have an equation of the form $(S_0, S_4') = 0$ which we can tackle in the same way as the lower-order master equation. There is a theoretical possibility that the terms such as the one we have just been discussing could be cohomologically non-trivial; this would then represent an obstruction to a given term being consistent at higher orders. If this situation were to arise the term in question would presumably have to be excluded.

We therefore see that the second- and third-order on-shell invariants do not require any correction terms, although there will be corrections to the supersymmetry transformations which can in principle be determined systematically, while from the fourth order up the lower-order invariants will induce higher-order corrections. Indeed, as we have seen, there is no independent pure F^6 invariant in the theory and this indicates that it is induced by the F^4 term.

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